

Inverse square problem and $so(2,1)$ symmetry in noncommutative space

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We study the quantum mechanics of a system with inverse square potential in noncommutative space. Both the coordinates and momentums are considered to be noncommutative, which breaks the original $so(2,1)$ symmetry. The energy levels and eigenfunctions are obtained. The generators of the $so(2,1)$ algebra are also studied in noncommutative phase space and the commutators are calculated, which shows that the $so(2,1)$ algebra obtained in noncommutative space is not closed. However the commutative limit $\Theta, \bar{\Theta} \rightarrow 0$ for the algebra smoothly goes to the standard $so(2,1)$ algebra.

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Noncommutative quantum mechanics has received lot of attention in recent years [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22]. It was Snyder who introduced the concept of noncommutative space-time in his work [23, 24] in 1947, where electrodynamics is treated in noncommutative spacetime. It is well known that the configuration space of a quantum mechanical system, confined to lowest Landau level, behaves as phase space and thus becomes noncommutative. In quantum mechanics noncommutativity is a property of the phase space and this lead to the Heisenberg uncertainty relation $\Delta x \Delta p_x \geq \hbar/2$. Coordinate space noncommutativity is much explored and actively being studied nowadays in diverse fields [25, 26]. The list includes quantum mechanics [27, 28, 29, 30, 31, 32, 33, 34, 35, 36], quantum field theory [37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48], string theory [49, 50, 51, 52] etc.

However momentum space noncommutativity in addition to coordinate space noncommutativity seems to be not much explored although discussed recently in some literatures [53, 54]. It is however known that in background magnetic field, $\mathbf{B} = (B^1, B^2, B^3)$, the different components of a generalized momenta do not commute, $[P^i, P^j] = i\epsilon^{ijk}B^k$. Note that the momentum noncommutativity due to background magnetic field is governed by the electric charge of the corresponding quantum particle. On the other hand we introduce momentum noncommutativity which is independent of electric charge. In this article we thus discuss a system with inverse square potential [55, 56, 57, 58, 59, 60, 61] in noncommutative momentum space in addition to coordinate space noncommutativity. Inverse square potential is very important because of its large range of applicability. Starting from electron scattering in polar molecules [58] to scalar field dynamics in near horizon space of many black hole spacetimes can be described by Schrödinger eigenvalue equation with inverse square potential. It is interesting from the theoretical point of view also. It belongs to a class of interactions which have conformal symme-

try. Due to this symmetry the system does not have any scale and thus it does not possess any finite ground state. Usually this system can be made physical from the bound state point of view by a suitable self-adjoint extensions [62, 63, 64, 65, 66, 67, 68, 69, 70]. Then the system possesses a finite ground state, which breaks the scale symmetry. Symmetry breaking in the process of quantization is called anomaly and inverse square interaction is thus a simple realization of scaling anomaly. Renormalization [71] is another technique by which this inverse square potential can be treated.

In a recent paper [72] we studied the inverse square system in only noncommutative coordinate space. It was found that the original $so(2,1)$ symmetry is broken by the scale Θ of the noncommutative space. The scale symmetry is explicitly broken by a potential of the form $\sim 2\Theta\alpha r^{-4}L_z$ (to lowest order in noncommutativity), which is generated due to the noncommutativity of the coordinate space. This system then possesses a bound state at threshold [73, 74, 75], i.e., $E = 0$. The system was also studied for large noncommutative parameter Θ and the bound state solutions are obtained, which however does not have the commutative limit.

The article is organized in the following fashion: we first briefly review the quantum mechanical system with inverse square potential in a noncommutative plane, which will set the platform for our next discussion. Then we come to the discussion of our present article and study the same system in noncommutative momentum space and study its solution and symmetry algebra. We also discuss the system where both the coordinate-coordinate and momentum-momentum noncommutativity is taken into account at a time. Finally we conclude.

Quantum mechanical system defined by the Hamiltonian H with the potential $V = \alpha r^{-2}$ is considered on noncommutative plane. In noncommutative plane, the standard algebra on phase space gets modified (we use the unit $\hbar = 1$)

$$\begin{aligned} [\bar{x}_i, \bar{x}_j] &= 2i\epsilon_{ij}\Theta, \quad [\bar{p}_i, \bar{p}_j] = 0, \quad [\bar{x}_i, \bar{p}_j] = i\delta_{ij}, \\ \epsilon_{12} &= -\epsilon_{21} = 1, \quad \epsilon_{11} = \epsilon_{22} = 0. \end{aligned} \quad (1)$$

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However, in the commutative limit $\Theta \rightarrow 0$, the algebra

(1) reduces to the well known algebra on phase space

$$[x_i, x_j] = 0, \quad [p_i, p_j] = 0, \quad [x_i, p_j] = i\delta_{ij}. \quad (2)$$

One possible realization for the noncommutative phase space coordinates $(\bar{x}_i, \bar{p}_i, i = 1, 2)$ in terms of standard coordinates $(x_i, p_i, i = 1, 2)$ is the following

$$\begin{aligned} \bar{x}_1 &= x_1 - \Theta p_2, & \bar{x}_2 &= x_2 + \Theta p_1, \\ \bar{p}_1 &= p_1, & \bar{p}_2 &= p_2. \end{aligned} \quad (3)$$

It can be easily checked that the representation (3) is consistent with the algebra (1) and (2). Because of the scale invariant potential V , the Hamiltonian H possesses $so(2, 1)$ symmetry

$$[D, H] = -iH, \quad [D, K] = iK, \quad [H, K] = 2iD, \quad (4)$$

in commutative plane $(x_i, i = 1, 2)$. Here D is the dilatation and K is the conformal operator. But when we consider the system in noncommutative plane $(\bar{x}_i, i = 1, 2)$, the scenario changes completely. The $so(2, 1)$ algebra up to first order in Θ is modified like [72]

$$\begin{aligned} [D_\Theta, H_\Theta] &= -iH_\Theta + \Theta\Delta_1, & [D_\Theta, K_\Theta] &= iK_\Theta + \Theta\Delta_2, \\ [K_\Theta, H_\Theta] &= -2iD_\Theta + \Theta\Delta_3, \end{aligned} \quad (5)$$

where $\Delta_1 = -2i\alpha r^{-4}L_z$, $\Delta_2 = (2i\alpha t^2 r^{-4} + i/2)L_z$, $\Delta_3 = -4i\alpha t r^{-4}L_z$. The generators up to first order in Θ can be written as [72]

$$H_\Theta = H + \Theta\Delta H, \quad D_\Theta = D + \Theta\Delta D, \quad K_\Theta = K + \Theta\Delta K, \quad (6)$$

where $\Delta H = 2\alpha r^{-4}L_z$, $\Delta D = 2\alpha r^{-4}L_z t$ and $\Delta K = (2\alpha t^2 r^{-4} - 1/2)L_z$. The above algebra (5), which is not closed, reduces to the well known $so(2, 1)$ algebra (4) in the limit $\Theta \rightarrow 0$.

We now come to the discussion of the system H in noncommutative momentum space. Since this time the momentums are noncommutative but coordinates are commutative the algebra for the phase space becomes

$$[\bar{x}_i, \bar{x}_j] = 0, \quad [\bar{p}_i, \bar{p}_j] = 2i\epsilon_{ij}\bar{\Theta}, \quad [\bar{x}_i, \bar{p}_j] = i\delta_{ij}. \quad (7)$$

Here the strength of noncommutativity in momentum space is given by the parameter $\bar{\Theta}$. The commutative limit $\bar{\Theta} \rightarrow 0$ of the algebra (7) goes to the standard result (2). We choose one possible realization for the noncommutative phase space coordinates $(\bar{x}_i, \bar{p}_i, i = 1, 2)$ in terms of standard coordinates $(x_i, p_i, i = 1, 2)$ as

$$\begin{aligned} \bar{x}_1 &= x_1, & \bar{x}_2 &= x_2, \\ \bar{p}_1 &= p_1 + \bar{\Theta}x_2, & \bar{p}_2 &= p_2 - \bar{\Theta}x_1. \end{aligned} \quad (8)$$

Taking the representation (8) the Hamiltonian $H = \mathbf{p}^2 + \alpha \mathbf{r}^{-2}$ can be written in noncommutative momentum space as

$$\begin{aligned} H_{\bar{\Theta}} &= \bar{\mathbf{p}}^2 + \alpha \bar{\mathbf{r}}^{-2} \\ &= \mathbf{p}^2 + \alpha \mathbf{r}^{-2} + \bar{\Theta}^2 \mathbf{r}^2 - 2\bar{\Theta}L_z. \end{aligned} \quad (9)$$

Once we write the Hamiltonian $H_{\bar{\Theta}}$ in terms of the standard coordinates, we can solve the eigenvalue problem

$$H_{\bar{\Theta}}\psi = E_{\bar{\Theta}}\psi. \quad (10)$$

This eigenvalue problem (10) can be solved exactly for different ranges of the coupling constant α , following a method of Ref. [70]. The problem can be analyzed in different coupling constant ranges $\alpha \geq 1 - m^2$, $-m^2 < \alpha < 1 - m^2$, $\alpha < -m^2$ and $\alpha = -m^2$. It is well known that for $\alpha \geq 1 - m^2$, the Hamiltonian $H_{\bar{\Theta}}$ is essentially self-adjoint and has unique self-adjoint extensions. The bound state solutions and eigenvalues are

$$E_{\bar{\Theta}}^{n,m} = 2\bar{\Theta} \left[2n - m + \sqrt{m^2 + \alpha + 1} \right], \quad (11)$$

$$R_{\bar{\Theta}}^{n,m}(r) = r^{\sqrt{\alpha+m^2}} e^{-\frac{1}{2}\bar{\Theta}r^2} L_n^{\sqrt{\alpha+m^2}}(\bar{\Theta}r^2), \quad (12)$$

where $L_n^{\sqrt{\alpha+m^2}}$ is Laguerre polynomial, $n = 0, 1, 2, \dots$ and $m = 0, \pm 1, \pm 2, \dots$. The emergence of these bound states with eigenvalues $E_{\bar{\Theta}}^{n,m}$ are the consequences of the breaking of $so(2, 1)$ symmetry due to the scale $\bar{\Theta}$. For $-m^2 < \alpha < 1 - m^2$, there is a one parameter family of self-adjoint extensions. The exact solution for a fixed self-adjoint extension parameter can be evaluated numerically. But for some special values of the extension parameter analytical results can be obtained also [70]. Among the two analytical results one coincides with the results (11) and (12) and the other is given by

$$E_{\bar{\Theta}}^{n,m} = 2\bar{\Theta} \left[2n - m - \sqrt{m^2 + \alpha + 1} \right], \quad (13)$$

$$R_{\bar{\Theta}}^{n,m}(r) = r^{-\sqrt{\alpha+m^2}} e^{-\frac{1}{2}\bar{\Theta}r^2} L_n^{-\sqrt{\alpha+m^2}}(\bar{\Theta}r^2). \quad (14)$$

It should be noted that despite the constraint $0 < \alpha + m^2 < 1$, all energy levels may not be positive here unlike the case of Ref. [70] where all energy levels for the same constraint take positive values. Depending upon the values of m , the energy levels (13) associated with high angular momentum may also take negative values. For $\alpha < -m^2$ the energy levels are unbounded from below and for $\alpha = -m^2$ the discussion for the eigenvalue and eigenfunction can also be discussed in line with Refs. [55, 56, 57].

It would be informative to study the $so(2, 1)$ algebra in noncommutative momentum space. To the first order of the noncommutativity $\bar{\Theta}$ the algebra becomes

$$[D_{\bar{\Theta}}, H_{\bar{\Theta}}] = -iH, \quad [D_{\bar{\Theta}}, K_{\bar{\Theta}}] = iK, \quad [K_{\bar{\Theta}}, H_{\bar{\Theta}}] = -2iD \quad (15)$$

which is easily seen to be reduced to the standard $so(2, 1)$ algebra (4) in the limit $\bar{\Theta} \rightarrow 0$.

We now consider the phase space where both the position space and momentum space are taken to be noncommutative. The noncommutative algebra on the phase space, obtained from (1) and (7), can be written as

$$[\bar{x}_i, \bar{x}_j] = 2i\epsilon_{ij}\bar{\Theta}, \quad [\bar{p}_i, \bar{p}_j] = 2i\epsilon_{ij}\bar{\Theta}, \quad [\bar{x}_i, \bar{p}_j] = i\delta_{ij}. \quad (16)$$

It is possible to get the noncommutative phase space coordinates in terms of standard coordinates. From (3) and (8) we get the following representation

$$\begin{aligned}\overline{x}_1 &= x_1 - \Theta p_2, & \overline{x}_2 &= x_2 + \Theta p_1, \\ \overline{p}_1 &= p_1 + \overline{\Theta} x_2, & \overline{p}_2 &= p_2 - \overline{\Theta} x_1,\end{aligned}\quad (17)$$

which is consistent with the algebra (16) up to the first order in noncommutative parameters Θ and $\overline{\Theta}$. Note that the representation (17) satisfies the first two commutators of the algebra (16) to all orders in noncommutative parameters Θ and $\overline{\Theta}$. But the last commutator is only satisfied for the first order in noncommutativity. One can also consider the all higher orders of the noncommutative parameters, which will give [53] (we explicitly keep \hbar here)

$$[\overline{x}_i, \overline{p}_j] = i\delta_{ij} [1 + \Theta\overline{\Theta}] \hbar. \quad (18)$$

Note that if we ignore terms higher than the first order in noncommutative parameters then the second term within the bracket in right side of the equation (18) should be ignored and the resulting equation is the standard expression. The fact that the Plank constant gets modified as [53, 76]

$$\overline{\hbar} = [1 + \Theta\overline{\Theta}] \hbar, \quad (19)$$

is a exclusive feature of the noncommutative phase space, which was absent both in noncommutative coordinate space and noncommutative momentum space. In Ref. [53] this effective Plank constant has been exploited to calculate a bound on the noncommutative parameters.

We now concentrate on the system in noncommutative phase space. The Hamiltonian in this situation becomes

$$\begin{aligned}H_{\Theta, \overline{\Theta}} &= \overline{p}^2 + \alpha \overline{r}^{-2} \\ &= \mathbf{p}^2 + \overline{\Theta}^2 \mathbf{r}^2 - 2\overline{\Theta} L_z + \frac{\alpha}{(\Theta^2 \mathbf{p}^2 + \mathbf{r}^2 - 2\Theta L_z)}.\end{aligned}\quad (20)$$

This Hamiltonian can be solved for large Θ using an algebraic method followed in Ref. [72]. For the moment we consider the denominator of the last term of the Hamiltonian $H_{\Theta, \overline{\Theta}}$,

$$\overline{H}_{\Theta} = \Theta^2 \mathbf{p}^2 + \mathbf{r}^2 - 2\Theta L_z, \quad (21)$$

and write it in terms of the Schwinger representation. The annihilation operators [33]

$$\begin{aligned}\overline{a}_+ &= (x_1 - ix_2) + \Theta(ip_1 + p_2), \\ \overline{a}_- &= (ix_1 - x_2) - \Theta(p_1 + ip_2),\end{aligned}\quad (22)$$

and its corresponding creation operators satisfy the commutation relation

$$[\overline{a}_+, \overline{a}_+^\dagger] = [\overline{a}_-, \overline{a}_-^\dagger] = 4\Theta. \quad (23)$$

Rest of the commutators are zero. The number operators can now be constructed as

$$\overline{n}_+ = \overline{a}_+^\dagger \overline{a}_+, \quad \overline{n}_- = \overline{a}_-^\dagger \overline{a}_-, \quad (24)$$

which satisfy the eigenvalue equation

$$\begin{aligned}\overline{n}_+ |n_+, n_-\rangle &= n_+ |n_+, n_-\rangle, n_+ = 0, 4\Theta, 8\Theta, 12\Theta, \dots \\ \overline{n}_- |n_+, n_-\rangle &= n_- |n_+, n_-\rangle, n_- = 0, 4\Theta, 8\Theta, 12\Theta, \dots\end{aligned}\quad (25)$$

The Hamiltonian \overline{H}_{Θ} can now be written in terms of the number operators as,

$$\overline{H}_{\Theta} = \overline{n}_- + 2\Theta, \quad (26)$$

which satisfy the equation

$$\overline{H}_{\Theta} |n_+, n_-\rangle = \overline{E}_{\Theta} |n_+, n_-\rangle \quad (27)$$

$$\overline{E}_{\Theta} = n_- + 2\Theta. \quad (28)$$

Now the eigenvalue of the Hamiltonian $H_{\Theta, \overline{\Theta}}$ in $|n_+, n_-\rangle$ basis becomes

$$\begin{aligned}E_{\Theta, \overline{\Theta}} &= \langle n_+, n_- | \mathbf{p}^2 + \overline{\Theta}^2 \mathbf{r}^2 - 2\overline{\Theta} L_z | n_+, n_- \rangle + \frac{\alpha}{n_- + 2\Theta} \\ &= \frac{1}{4\Theta^2} [n_+ + n_- + 4\Theta] + \frac{\overline{\Theta}^2}{4} [n_+ + n_- + 4\Theta] \\ &\quad - \frac{\overline{\Theta}}{2\Theta} [n_+ - n_-] + \frac{\alpha}{n_- + 2\Theta}.\end{aligned}\quad (29)$$

One should note that although the limit $\Theta \rightarrow 0$ can not be taken in (29) directly but the limit $\overline{\Theta} \rightarrow 0$ can be taken smoothly and the limit goes to the result, Eq. (24), of Ref. [72]. It is obvious that the existence of the eigenvalue (29) is a consequence of the breaking of the $so(2, 1)$ symmetry. It is therefore interesting to get an explicit commutator relations of the three generators of the $so(2, 1)$ algebra in noncommutative phase space, which is up to first order in noncommutative parameters Θ and $\overline{\Theta}$,

$$[D_{\Theta, \overline{\Theta}}, H_{\Theta, \overline{\Theta}}] = -iH_{\Theta} + \Theta\Delta_1, \quad (30)$$

$$[D_{\Theta, \overline{\Theta}}, K_{\Theta, \overline{\Theta}}] = iK_{\Theta} + \Theta\Delta_2, \quad (31)$$

$$[K_{\Theta, \overline{\Theta}}, H_{\Theta, \overline{\Theta}}] = -2iD_{\Theta} + \Theta\Delta_3. \quad (32)$$

Note that the right hand side depends only on the coordinate space noncommutative parameter Θ , where we consider only first order terms. Note also that the commutative limit $\Theta, \overline{\Theta} \rightarrow 0$ goes to the standard $so(2, 1)$ algebra.

In conclusion, the inverse square potential is studied in noncommutative space. Both the coordinates and momentums are considered to be noncommutative. The bound state solutions are obtained, which is a consequences of the scale symmetry breaking by the noncommutative parameters Θ and $\overline{\Theta}$. The three generators of

the $so(2,1)$ algebra are also studied and the commutators are constructed to first order in noncommutativity, which shows that the algebra is not closed. However the

algebra reduces in the commutative limit $\Theta, \bar{\Theta} \rightarrow 0$ to standard $so(2,1)$ algebra.

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